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Effects of Inhomogeneities in Atmospheric Turbulence on the Dynamic Response of an Aircraft

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The power-spectral density of the dynamic response of an aircraft to inhomogeneous turbulence is formulated. In order to facilitate the analysis of the response the inhomogeneous turbulence is decomposed into two parts—a part whose statistics, viz. the scale and the intensity, are spatially constant and a part whose statistics are spatially varying. The effects of the inhomogeneities on the response are analyzed through a "correction factor" whose numerical value is a measure of the percent error introduced into the response power-spectrum whenever the effects of the inhomogeneities are neglected. Numerical results are obtained for the cases of an airplane flying through turbulence whose intensity varies sinusoidally in space and an airplane flying through turbulence whose integral scale varies sinusoidally in space.

Nomenclature

$h(r,\tau)$	= space-dependent impulse response function of aircraft
$H(k,\omega)$	= generalized frequency response function of aircraft
k	= wave-number vector
$q(t)$	= response parameter of aircraft
$Q(x,r,\tau)$	= "normalized" correlation function of inhomogeneous turbulence
r_1,r_2	= position vectors $w \cdot r \cdot t$, a fixed reference frame
$w(x,t)$	= space- and time-dependent turbulent downwash
α	= parameter describing wavelength of intensity variations
β	= parameter describing wavelength of scale variations
$\delta(\cdot)$	= Dirac function
κ	= wave-number vector
Λ	= turbulence scale
σ	= turbulence intensity
$\Delta(\omega)$	= correction factor
$\phi_q(\omega)$	= power-spectrum of aircraft response
$\Phi(k,\omega)$	= power-spectrum of three-dimensional homogeneous turbulence
$\psi(x,r,\tau)$	= correlation function of turbulent downwash
$\Psi(\kappa,k,\omega)$	= generalized power-spectrum of inhomogeneous turbulence

I. Introduction

THE determination of the effects of inhomogeneities in atmospheric turbulence on the dynamic response of an aircraft is a problem that has not received much attention in the literature. Classical aircraft response analyses have dealt only with turbulence which is homogeneous, isotropic, and stationary (see Ref. 1 and references listed therein). More recently[2-6] the effects of nonstationary turbulence on the response of an aircraft have been modeled but the turbulence was still considered as homogeneous and isotropic.

In the sequel the effects of the nonhomogeneous, and therefore *nonisotropic,* nature of atmospheric turbulence on the response of an aircraft will be analytically determined and analyzed. The turbulence and the aircraft both will be considered as three-dimensional for the general response problem, although for specific numerical results only one direction of inhomogeneity will be necessary.

II. Formulation of General Response Problem

The dynamic response of an aircraft to atmospheric turbulence, in any response parameter q, can be written as a function of time as†

$$q(t) = \int h(r,\tau)\, w(r,t-\tau)\, \mathrm{d}r\mathrm{d}\tau \tag{1}$$

In this equation $w(\cdot)$ is the downwash turbulence velocity component of zero mean value, $\mathrm{d}r = \mathrm{d}x\mathrm{d}y\mathrm{d}z$, and $h(\cdot)$ is the space-dependent impulse-response function to the aircraft uniquely defining the response of the aircraft in q, at time t, to a unit impulse of downwash applied to the aircraft structure at the space-time point $(r, t-\tau)$. Employing the methods of multidimensional Fourier analysis[7] Eq. (1) leads to the power spectrum of the response, $\phi_q(\omega)$, which may be expressed as

$$\phi_q(\omega) = \left(\frac{1}{2\pi}\right)^6 \int H^*(k+\kappa/2,\omega)\, H(k-\kappa/2,\omega)\, \Psi(\kappa,k,\omega)\, \mathrm{d}\kappa\mathrm{d}k \tag{2}$$

where $\Psi(\cdot)$ is the generalized power-spectral density of the inhomogeneous turbulence downwash and is itself defined as the space-time Fourier transform

$$\Psi(\kappa,k,\omega) = \int \psi(x,r,\tau) \exp[\iota(\kappa\cdot x - k\cdot r - \omega\tau)]\, \mathrm{d}x\mathrm{d}r\mathrm{d}\tau \qquad (\iota = \sqrt{-1}) \tag{3}$$

of the two-point velocity correlation function

$$\psi(x,r,\tau) = \langle w(x-r/2,t)\, w(x+r/2,t+\tau) \rangle \tag{4}$$

while $H(\cdot)$ is the generalized frequency response function of the aircraft, also defined as a space-time Fourier transform

$$H(k,\omega) = \int h(r,\tau) \exp[\iota(k\cdot r - \omega\tau)]\, \mathrm{d}r\mathrm{d}\tau \tag{5}$$

and $\mathrm{d}\kappa = \mathrm{d}\kappa_x \mathrm{d}\kappa_y \mathrm{d}\kappa_z$, $\mathrm{d}k = \mathrm{d}k_x \mathrm{d}k_y \mathrm{d}k_z$. The symbol * in Eq. (2) denotes complex conjugate, while the symbol $\langle\ \rangle$ in Eq. (4) denotes mean value. The vectors x and r in Eq. (4) are defined

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†Unless stated otherwise, the limits on all integrals are $-\infty$ to $+\infty$.

respectively as $(r_2+r_1)/2$ and r_2-r_1, where r_1 and r_2 represent two points in space measured with respect to a fixed reference frame.

In order to attach some physical significance to the power spectrum of the response, as given by Eq. (2), it is necessary to point out that $\Psi(\kappa,k,\omega)$ may have both real and imaginary parts but that since $\phi_q(\cdot)$ is a real quantity only the real part of the right-hand side of Eq. (2) may be considered as the spectral quantity in question. Furthermore, as it may be shown by substitution into Eq. (1), $H(k,\omega)$ represents the complex amplitude of the response of the aircraft to the fundamental loading

$$w(r,t)=\exp[\iota(k\cdot r+\omega t)] \tag{6}$$

a loading which can be described as a "washboard wave pattern," of unit amplitude and wavelength $\lambda=2\pi/|k|$, moving in the direction $-k/|k|$ at the speed $V=\omega/|k|$, hence $H(k\pm\kappa/2,\omega)$ represents the complex amplitude of the response of the aircraft to the "modified" wave pattern

$$w(r,t)=\exp[\iota(k\cdot r\pm\kappa\cdot r/2+\omega t)] \tag{7}$$

Together with this, it follows from Eq. (3) that

$$\psi(x,r,\tau)=\left(\frac{1}{2\pi}\right)^7\int\exp[-\iota(\kappa\cdot x-k\cdot r-\omega\tau)]\Psi(\kappa,k,\omega)\,\mathrm{d}\kappa\mathrm{d}k\mathrm{d}\omega \tag{8}$$

which suggest that $\Psi(\cdot)$ is closely related to the Fourier amplitudes of the modified wave patterns. Accordingly, $\phi_q(\cdot)$ may be characterized as the sum of the respective energies transferred to the aircraft at various particular frequencies ω by each of the modified wave patterns of which the turbulence is composed. Note that Eq. (2) takes into account not only the effects of the distribution over all space of the turbulence downwash but also the effects of the three-dimensional nature of the aircraft structure. The effects of the inhomogeneities are manifested through the dependence of Ψ on the wave-number vector κ; indeed, when the turbulence is homogeneous rather than inhomogeneous, $\Psi(\cdot)$ is expressed as

$$\Psi^{(1)}(\kappa,k,\omega)=(2\pi)^3\delta(\kappa)\Phi^{(1)}(k,\omega) \tag{9}$$

where $\delta(\cdot)$ is the Dirac function, and Eq. (2) reduces to

$$\phi_q^{(1)}(\omega)=\left(\frac{1}{2\pi}\right)^3\int|H(k,\omega)|^2\Phi^{(1)}(k,\omega)\,\mathrm{d}k \tag{10}$$

The solution to Eq. (2) is more difficult than the solution to its homogeneous counterpart, Eq. (10).‡ The reason for this is that the functional form of $\psi(\cdot)$, and therefore of its Fourier transform $\Psi(\cdot)$, for inhomogeneous turbulence is not known, and there does not appear to exist in the literature a comprehensive theory from which the functional form of the correlation function can be determined in the general case. It is therefore necessary to make some sort of approximation to the functional form of $\psi(\cdot)$ for the specific types of turbulence which may be encountered by an aircraft flying through the atmosphere, viz., spatial changes in the turbulence intensity and spatial changes in the turbulence scale. Both of these types of inhomogeneities may be encountered by an aircraft as it travels from one turbulence "patch" to another, or as the aircraft encounters turbulence from one day to the next.

‡Method for approximating $H(k,\omega)$ and thus solving Eq. (10) for the case when the aircraft is "large" is found in Ref. 8. Methods for solving Eq. (10) when the aircraft is "small" may be found in the literature.

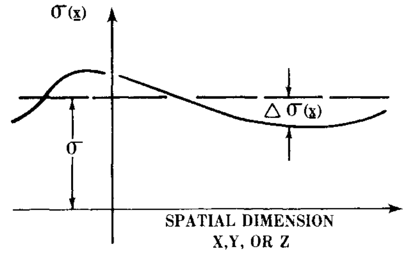

Fig. 1 Representation of spatially varying intensity as $\sigma(x)=\sigma+\Delta\sigma(x)$.

III. Representation of the Turbulence

In view of the foregoing, the correlation function for inhomogeneous turbulence is written as[9]

$$\psi(x,r,\tau)=\psi^{(1)}(r,\tau)+\psi^{(2)}(x,r,\tau) \tag{11}$$

where $\psi^{(1)}(\cdot)$ is independent of the position vector x and defines the correlation function of the "homogeneous" part of the turbulence, i.e., that part of the turbulence whose statistics are spatially constant, while $\psi^{(2)}(\cdot)$ is indeed dependent on x and defines the correlation function of the "inhomogeneous" part of the turbulence, i.e., that part of the turbulence whose statistics do vary in space. Such a representation of the turbulence is possible as a consequence of the fact that the correlation function of the downwash, Eq. (4), can, by virtue of the Schwartz inequality

$$\langle[Aw(x-r/2,t)+w(x+r/2,t+\tau)]^2\rangle\geq 0 \tag{12}$$

where A is an arbitrary constant, be expressed in the "separated" form

$$\psi(x,r,\tau)=\sigma(x-r/2)\sigma(x+r/2)Q(x,r,\tau) \tag{13}$$

where $\sigma(\cdot)$ is the intensity of the turbulence and $Q(\cdot)$ is the "normalized" correlation function satisfying the conditions $Q(x,r,\tau)\leq 1$ and $Q(x,0,0)=1$. In this separated form the effects on the correlation function, and therefore on the generalized power-spectral density, of spatial changes in the turbulence intensity and in the turbulence scale, $\Lambda(\cdot)$, manifest themselves, respectively, through the functions $\sigma(\cdot)$ and $Q(\cdot)$. For the particular case where the inhomogeneities are in the turbulence intensity only, i.e., the intensity only varies in space while the scale is spatially constant, it follows that (see Fig. 1)

$$\sigma(x)=\sigma+\Delta\sigma(x) \tag{14}$$

and that the normalized correlation function $Q(x,r,\tau)$ simplifies to $Q^{(1)}(r,\tau)$, i.e., a functional form which is independent of x. Substituting Eq. (14) into Eq. (13) provides the required form of $\psi(x,r,\tau)$, Eq. (11), where $\psi^{(1)}(r,\tau)=\sigma^2Q^{(1)}(r,\tau)$ and $\psi^{(2)}(x,r,\tau)=\sigma[\Delta\sigma(x-r/2)+\Delta\sigma(x+r/2)]Q^{(1)}(r,\tau)$. A similar procedure can be pursued in the case where the inhomogeneities are in the turbulence scale only, i.e., the intensity is constant while the scale is spatially varying, by first writing

$$\Lambda(x)=\Lambda+\Delta\Lambda(x) \tag{15}$$

while simultaneously expressing the function $Q(\cdot)$ as[9]

$$Q(x,r,\tau)=Q^{(1)}(r,\tau)+Q^{(2)}(x,r,\tau) \tag{16}$$

The presence of the position vector x in the function $Q^{(2)}(\cdot)$ indicates that this function describes the spatial variations in the scale while the absence of x in the function $Q^{(1)}(\cdot)$ denotes that this function describes the spatially constant part of the scale. The resulting expression for the correlation function is

$$\psi(x,r,\tau) = \sigma^2 Q^{(1)}(r,\tau) + \sigma^2 Q^{(2)}(x,r,\tau) \tag{17}$$

For turbulence which is inhomogeneous in both the scale and the intensity, the appropriate expression for the correlation function is

$$\begin{aligned}\psi(x,r,\tau) &= \sigma^2 Q^{(1)}(r,\tau) + \sigma^2 Q^{(2)}(x,r,\tau) \\ &+ \sigma[\Delta\sigma(x-r/2) + \Delta\sigma(x+r/2)][Q^{(1)}(r,\tau) \\ &+ Q^{(2)}(x,r,\tau)] + \Delta\sigma(x-r/2)\Delta\sigma(x+r/2) \\ &\times [Q^{(1)}(r,\tau) + Q^{(2)}(x,r,\tau)]\end{aligned} \tag{18}$$

An interesting feature of this type of turbulence is the presence of the effects on the correlation function of scale-intensity coupling, represented in Eq. (18) by the terms involving the product of $\Delta\sigma(\cdot)$ and $Q^{(2)}(\cdot)$. These effects, however, are negligible whenever the changes in the intensity and the changes in the scale are both "small," in which case the turbulence may be described as "almost homogeneous." The correlation function corresponding to this latter type of turbulence is

$$\begin{aligned}\psi(x,r,\tau) &= \sigma^2 [Q^{(1)}(r,\tau) + Q^{(2)}(x,r,\tau)] \\ &+ \sigma[\Delta\sigma(x-r/2) + \Delta\sigma(x+r/2)] Q^{(1)}(r,\tau)\end{aligned} \tag{19}$$

IV. Effects of Inhomogeneities on the Response of the Aircraft

The results of Sec. III suggest that the generalized power-spectral density of the turbulence can also be decomposed into the form

$$\Psi(\kappa,k,\omega) = \Psi^{(1)}(\kappa,k,\omega) + \Psi^{(2)}(\kappa,k,\omega) \tag{20}$$

where the definitions of $\Psi^{(1)}(\cdot)$ and $\Psi^{(2)}(\cdot)$ are obvious. Furthermore, the power spectrum of the response now becomes

$$\phi_q(\omega) = \phi_q^{(1)}(\omega) + \phi_q^{(2)}(\omega) \tag{21}$$

where $\phi_q^{(1)}(\omega)$ is defined in Eq. (10) and

$$\begin{aligned}\phi_q^{(2)}(\omega) &= (1/2\pi)^6 \int H^*(k+\kappa/2,\omega) \\ &\times H(k-\kappa/2,\omega)\Psi^{(2)}(\kappa,k,\omega)\,d\kappa dk\end{aligned} \tag{22}$$

Equation (21) indicates that the effects of the spatial variations in the turbulence statistics may, in the determination of the power spectrum of the response, be considered independently. Indeed, if Eq. (21) is rewritten as

$$\phi_q(\omega) = \phi_q^{(1)}(\omega)[1 + \Delta(\omega)] \tag{23}$$

where

$$\Delta(\omega) = \phi_q^{(2)}(\omega)/\phi_q^{(1)}(\omega) \tag{24}$$

is a correction term, it then follows that the effects of the inhomogeneities on the power spectrum of the response can be assessed entirely through the evaluation of $\Delta(\omega)$. To be sure, whenever $\Delta(\omega) < 0$ it may then be concluded that the determination of $\phi_q(\omega)$ under the assumption of homogeneous turbulence, i.e., the determination of $\phi_q(\omega)$ with the tacit assumption that the effects of the inhomogeneous part of the turbulence are completely negligible, will consequently lead to a "conservative" estimate, i.e., an over-estimation, of $\phi_q(\omega)$, while on the other hand if $\Delta(\omega) > 0$ the determination of $\phi_q(\omega)$ under the assumption of homogeneous turbulence will clearly lead to a "nonconservative" estimate, i.e., an under-estimation, of $\phi_q(\omega)$. From the standpoint of the design of an aircraft with respect to fatigue and service life, the former case is the desirable one, since for this case the use of $\phi_q(\omega)$ to determine the mean-square response (or some other preliminary design parameter) of the aircraft will result in "over-designing" for the effects of turbulence while the latter case will likely lead to an "under-design." For the circumstance where $\Delta(\omega) = 0$, the effects of the inhomogeneities are unimportant. As for the effects of aircraft size on $\Delta(\omega)$, note that even when the aircraft is "small," i.e., even when the largest dimension of the aircraft is small in comparison to the constant part of the turbulence scale, the inhomogeneous nature of the turbulence must still be considered. This result follows by setting the wave-number vector $\boldsymbol{k}$, in the function $H(\cdot)$, identical to zero, hence reducing Eq. (2) to the form

$$\phi_q(\omega) = \left(\frac{1}{2\pi}\right)^6 \int H^*(\kappa/2,\omega) H(-\kappa/2,\omega)\Psi(\kappa,k,\omega)\,d\kappa dk \tag{25}$$

The presence of κ in the functional form of $H(\cdot)$ is evidence that $\Delta(\omega)$ will not necessarily be zero for this case and therefore that the inhomogeneous nature of the turbulence still needs to be considered in the determination of the power spectrum of the response.

V. Illustrative Example

Consider the problem of determining the power spectrum, $\phi_{\ddot z}(\omega)$, of the rigid-body normal acceleration, $\ddot z$, of an airplane whose dimensions are small in comparison to the integral scale of the turbulence. To determine the effects of intensity variations only on the power spectrum of the response the model of inhomogeneous turbulence to be used is turbulence whose intensity varies, in the x direction only (see Fig. 2), as

$$\sigma(x) = \sigma + \Delta\sigma \exp(\iota\alpha x), \qquad (\Delta\sigma/\sigma) \ll 1 \tag{26}$$

Accordingly, the generalized power spectrum of the input, when expressed with respect to a reference frame that is moving with, but not rigidly attached to, the aircraft is[9,10]

$$\begin{aligned}\Psi(\kappa,k,\omega) &= \{\phi^{(1)}(k)2\pi\delta(\kappa) + (\Delta\sigma/\sigma)2\pi\delta(\kappa+\alpha) \\ &\times [\phi^{(1)}(k+\alpha/2) + \phi^{(1)}(k-\alpha/2)]\}2\pi\delta(\omega - kU + \kappa U/2)\end{aligned} \tag{27}$$

where $\phi^{(1)}(k)$ is the spectrum of homogeneous *isotropic* turbulence and is given in the literature as

$$\phi^{(1)}(k) = 2\Lambda\sigma^2[1 + 3(k\Lambda)^2]/[1 + (k\Lambda)^2]^2 \tag{28}$$

The corresponding power spectra, $\phi_{\ddot z}^{(1)}(\omega)$ and $\phi_{\ddot z}^{(2)}(\omega)$, are

$$\phi_{\ddot z}^{(1)}(\omega) = |H(\omega)|^2 \phi^{(1)}\left(\frac{\omega}{U}\right) \tag{29}$$

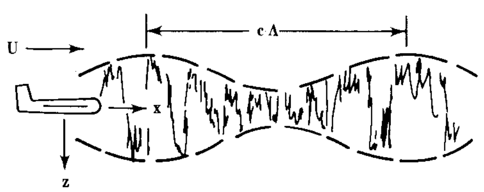

Fig. 2 Aircraft flying through turbulence of sinusoidally varying intensity, i.e., $\Delta\sigma(x) = \exp(\iota\alpha x)$, $\alpha = 2\pi/c\Lambda$.

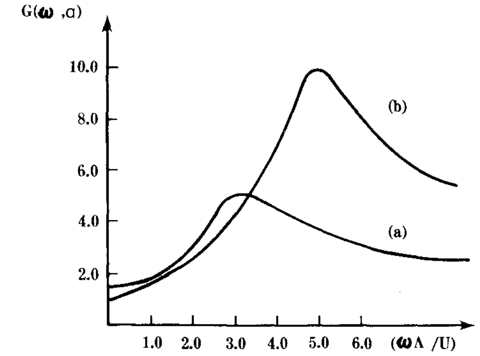

Fig. 3 Plot of $G(\omega,\alpha)$ for $\Lambda = 10^3$ ft, $\alpha = 0.002$ (curve a), and $\Lambda = 10^3$ ft, $\alpha = 0.004$ (curve b).

and

$$\phi_{\ddot{z}}^{(2)}(\omega) = \left(\frac{\Delta\sigma}{\sigma}\right) H^*(-\alpha/2,\omega) H(\alpha/2,\omega)$$

$$\times \left[\phi^{(I)}\left(\frac{\omega}{U}\right) + \phi^{(I)}\left(\frac{\omega}{U} - \alpha\right)\right] \tag{30}$$

where $H(\omega)$ is the complex amplitude of the response of the aircraft to the sinusoidal gust $w(t) = \exp(\iota\omega t)$, and is proportional to Sears' function $S(\omega/U)$, while $H(\alpha/2,\omega)$ is the complex amplitude of the response of the aircraft to the modified gust $w_\alpha(t) = \exp[\iota(\omega + \alpha U/2)t]$, and is proportional to $S(\omega/U + \alpha/2)$. The correction factor expresses as

$$\Delta_{\ddot{z}}(\omega) = \left(\frac{\Delta\sigma}{\sigma}\right) F(\omega,\alpha)\, G(\omega,\alpha) \tag{31}$$

where

$$F(\omega,\alpha) = S^*\left(\frac{\omega}{U} - \frac{\alpha}{2}\right) S\left(\frac{\omega}{U} + \frac{\alpha}{2}\right) \Big/ \left| S\left(\frac{\omega}{U}\right)\right|^2 \tag{32}$$

and

$$G(\omega,\alpha) = I + \phi^{(I)}\left(\frac{\omega}{U} - \alpha\right) \Big/ \phi^{(I)}\left(\frac{\omega}{U}\right) \tag{33}$$

For the domain of (ω/U) of interest and for values of α which give $\sigma(x)$ physical realizability, e.g., $\alpha = 0.002$ and 0.004, it can be shown that $\mathrm{Re}[F(\omega,\alpha)] \simeq 1.00$ and that $\mathrm{Im}[F(\omega,\alpha)]$ is negligible so that the expression for the correction factor simplifies to

$$\Delta_{\ddot{z}}(\omega) \simeq \left(\frac{\Delta\sigma}{\sigma}\right) G(\omega,\alpha) \tag{34}$$

Figure 3 is a plot of $G(\omega,\alpha)$ for the denoted values of α. The selected values of α were determined by setting $\alpha = 2\pi/c\Lambda$, where c is a number that defines the wavelength of the sinusoidal variation in $\sigma(\cdot)$ (see Fig. 2), hence for a value of $\Lambda = 1000$ ft the value of $\alpha = 0.002$ corresponds to a value of $c = \pi$, defining a sinusoidal variation in $\sigma(\cdot)$ of wavelength approximately equal to 3000 ft; the value of $\alpha = 0.004$ similarly corresponds to a sinusoidal variation in $\sigma(\cdot)$ of wavelength approximately equal to 1600 ft.

The effects of scale variation on $\Delta_{\ddot{z}}(\omega)$, and therefore on $\phi_{\ddot{z}}(\omega)$, can be analyzed by choosing the model of turbulence to be such that

$$\Lambda(x) = \Lambda + \Delta\Lambda \exp(\iota\beta x), \qquad (\Delta\Lambda/\Lambda) \ll I \tag{35}$$

while the intensity is constant. The generalized power spec-

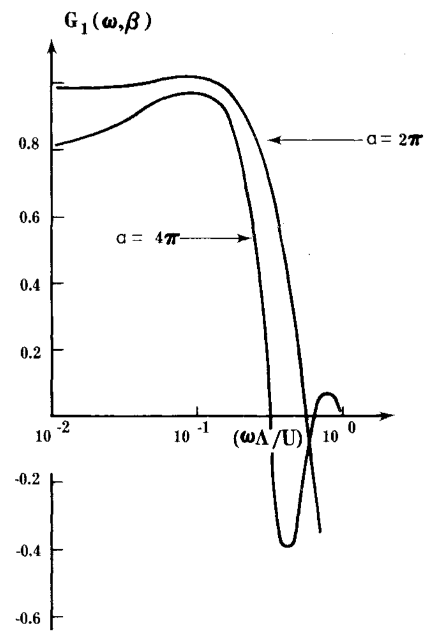

Fig. 4 Plot of $G_I(\omega,\beta)$ for $\Lambda = 10^3$ ft and $\beta = 0.0002$.

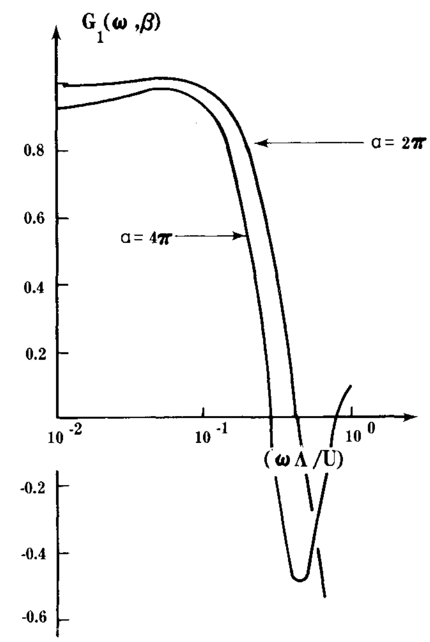

Fig. 5 Plot of $G_I(\omega,\beta)$ for $\Lambda = 10^3$ ft and $\beta = 0.0001$.

trum of the input is then

$$\Psi(\kappa,k,\omega) = [\phi^{(I)}(k) 2\pi\delta(\kappa) + (\Delta\Lambda/a\Lambda) 2\pi\delta(\kappa+\beta)$$

$$\times \phi^{(2)}(k)] 2\pi\delta(\omega - kU + \kappa U/2) \tag{36}$$

where[9]

$$\phi^{(2)}(k) = 2\Lambda\sigma^2\left(\frac{2\pi}{a}\right) \frac{\sin(ak\Lambda)}{(k\Lambda)[(2\pi/a)^2 - (k\Lambda)^2]^2} \tag{37}$$

and a is a parameter which defines the domain of definition of $\psi^{(2)}(x,r,\tau)$ in the variable r (see Ref. 9). The corresponding portion of the response power spectrum, $\phi_{\ddot{z}}^{(2)}(\omega)$, is

$$\phi_{\ddot{z}}^{(2)}(\omega) = \left(\frac{\Delta\Lambda}{a\Lambda}\right) H^*\left(\frac{-\beta}{2},\omega\right) H\left(\frac{\beta}{2},\omega\right) \phi^{(2)}\left(\frac{\omega}{U} - \frac{\beta}{2}\right) \tag{38}$$

and the correction factor is

$$\Delta_{\ddot{z}}(\omega) = \left(\frac{\Delta\Lambda}{a\Lambda}\right) F(\omega,\beta)\phi^{(2)}\left(\frac{\omega}{U} - \frac{\beta}{2}\right) \Big/ \phi^{(1)}\left(\frac{\omega}{U}\right) \tag{39}$$

Here, as before, the $\mathrm{Re}[F(\omega,\beta)] \simeq 1.00$ and the $\mathrm{Im}[F(\omega,\beta)]$ is negligible for the values of (ω/U) and β of interest. Therefore

$$\Delta_{\ddot{z}}(\omega) \simeq \left(\frac{\Delta\Lambda}{a\Lambda}\right)\phi^{(2)}\left(\frac{\omega}{U} - \frac{\beta}{2}\right) \Big/ \phi^{(1)}\left(\frac{\omega}{U}\right) \tag{40}$$

Figure 4 is a plot of

$$G_I(\omega,\beta) = \left(\frac{1}{a}\right)\phi^{(2)}\left(\frac{\omega}{U} - \frac{\beta}{2}\right)\phi^{(1)}\left(\frac{\omega}{U}\right) \tag{41}$$

for the values $a = 2\pi$ and 4π and the value of $\beta = 0.0002$; Fig. 5 is a plot of $G_I(\omega,\beta)$ for $a = 2\pi$, $a = 4\pi$ and the value of $\beta = 0.0001$. The values of β were selected by setting $\beta = 2\pi/b\Lambda$ so that $\beta = 0.0002$ corresponds to $\Lambda = 1000$ ft and $b = 10\pi$ defining a wavelength, of the oscillatory part of the scale, of approximately 6 miles. The value of $\beta = 0.0001$ defines a wavelength of approximately 12 miles.

VI. Discussion

The power-spectral density of the dynamic response of an aircraft to inhomogeneous atmospheric turbulence has been formulated in the general case where the turbulence and the aircraft are both three dimensional. The inclusion of the inhomogeneous effects is a necessary feature because there exists an abundance of data (see Ref. 11 and references listed therein) that atmospheric turbulence is indeed inhomogeneous, particularly in those turbulence statistics which directly affect the response of an aircraft, viz., turbulence intensity and turbulence scale. In the formulation the notion of a generalized power-spectral density for the turbulence is introduced and it is this parameter which contains the required information, through its dependence on $\boldsymbol{\kappa}$, about the inhomogeneous character of the turbulence; for $\boldsymbol{\kappa} \equiv 0$ the turbulence is homogeneous while for $\boldsymbol{\kappa} \neq 0$ the turbulence is inhomogeneous. It is suggested through the formulation that the power spectrum of the response can be interpreted as the sum of the respective energies transferred to the aircraft, at various particular frequencies ω, by each of the various wave patterns of which the turbulence is composed. The relationship between the power spectrum of the response when the turbulence is homogeneous and the power spectrum of the response when the turbulence is inhomogeneous is portrayed by comparing Eqs. (2) and (10). This relationship is used as a basis for decomposing the turbulence into a part whose statistics are spatially constant and a part whose statistics are spatially varying. As is shown in Secs. III and IV the resulting expression for the generalized spectral density facilitates the problem of determining the effects of the inhomogeneities in that these effects can be consequently analyzed entirely through a correction factor whose numerical value is a measure of the percent effort introduced into the response problem when the effects of particular well-defined inhomogeneities are neglected. It is shown that the effects of the inhomogeneities need to be considered even when the aircraft is "small," i.e., when the largest dimension of the aircraft is small in comparison to the turbulence scale. And while the numerical results presented in Sec. V are, in view of the specific spatial variation in both the scale and the intensity employed, highly specialized those results are nevertheless useful. Indeed, any general functional spatial variation in the scale or intensity can, through the process of Fourier analysis, be approximated as the finite sum of exponentials, i.e.,

$$\sigma(\boldsymbol{x}) \simeq \sum_{n=0}^{N} A_n \exp(\iota\boldsymbol{\alpha}_n \cdot \boldsymbol{x}) \tag{42}$$

$$\Lambda(\boldsymbol{x}) \simeq \sum_{m=0}^{M} B_m \exp(\iota\boldsymbol{\beta}_m \cdot \boldsymbol{x}) \tag{43}$$

so that the error introduced into the aircraft response problem by neglecting whatever functional variation in $\sigma(\cdot)$ or $\Lambda((\cdot)$ there may exist, can at least be "roughly" approximated by defining a priori the type of turbulence through which the aircraft is expected to fly. If the aircraft is expected to encounter "moderate-to-severe-to-light" large-scale turbulence on one day and "moderate-to-extreme-to-light" small-scale turbulence on the next, then that model of inhomogeneous turbulence should be used in the response analysis.

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